Algebraic structure of the two-qubit quantum Rabi model and its solvability using Bogoliubov operators

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Abstract. We have found the algebraic structure of the two-qubit quantum Rabi model behind the possibility of its novel quasi-exact solutions with finite photon numbers by analyzing the Hamiltonian in the photon number space. The quasi-exact eigenstates with at most 1 photon exist in the whole qubit-photon coupling regime with constant eigenenergy equal to single photon energy $\hbar\omega$, which can be clear demonstrated from the Hamiltonian structure. With similar method, we find these special "dark states"-like eigenstates commonly exist for the two-qubit Jaynes-Cummings model, with $E = N\hbar\omega$ ($N = -1, 0, 1, \ldots$), and one of them is also the eigenstate of the two-qubit quantum Rabi model, which may provide some interesting application in a simper way. Besides, using Bogoliubov operators, we analytically retrieve the solution of the general two-qubit quantum Rabi model. In this more concise and physical way, without using Bargmann space, we clearly see how the eigenvalues of the infinite-dimensional two-qubit quantum Rabi Hamiltonian are determined by convergent power series, so that the solution can reach arbitrary accuracy reasonably because of the convergence property.

1. Introduction

The quantum Rabi model [1] describes the interaction between a bosonic mode and a two–level system—probably the simplest interaction between light and matter. Its semiclassical form was first introduced by Rabi in nuclear magnetic resonance [2]. In 1963, Jaynes and Cummings [1] found its application in describing the interaction between a two-level molecular and a single mode photon field. With the developments of experiments, many systems can be described by this model in quantum optics [3], condensed matter [4], cavity quantum electrodynamics (QED) [5], circuit QED [6], quantum dots [7], trapped ions [8] and so on. Although this model takes a very simple form, its analytical solution was not so easy to obtain, so various approximations were made, one of which is the famous "rotating–wave approximation" [1]. In 2011, its solution was analytically found by Braak [9] in the Bargmann space [10]. It can describe the ultrastrong qubit-photon coupling regime, which has been reached in recent circuit QED experiments [11], where the "rotating wave approximation" breaks down. After that, various researches are done to the full Rabi Hamiltonian, including recovering the solution of the Rabi model [12–14], real-time dynamics [15], the solution of the two-qubit Rabi Hamiltonian [16–20], dynamical correlation functions [21], and so on [22–31].

Two-qubit system is basic and fundamental to the construction of the universal quantum gate. Various qubit-qubit interactions are applied to generate qubit-qubit entanglement and realize quantum computation [32,33], one of which is mediated by a resonant cavity, described by the two-qubit quantum Rabi model [19]. In this case, the ultrafast two-qubit quantum gate can be constructed in the ultrafast qubit-photon coupling regime [34]. Besides, the distant qubits can be coupled through a resonant cavity and the coherent quantum state storage and transfer can be realized [35]. Working for the whole qubit-photon coupling regime, the two-qubit quantum Rabi model can be applied in many systems in quantum optics [36] and quantum information [37]. Its analytical solution was obtained in [19] by means of Bargmann space approach, and also in [20] with extended coherent states representation. One interesting result is that there exist coupling-dependent eigenstates in the whole coupling regime with constant eigenenergy–reminiscent of "dark states", but they are coupling-dependent and the photon number is bounded from above at 1, which is novel and interesting. Besides, there are quasi-exact solutions with finite photon numbers N, which are not presented in the one-qubit Rabi model. These special solutions may have some interesting application, however, the algebraic structure behind the possibility of these special solutions needs to be clarified.

In this paper, we have clarified the algebraic structure of the two-qubit quantum Rabi model for its special quasi-exact solutions with finite photon numbers found in [19]. By analyzing the Hamiltonian in the photon number space, we find the condition for closed subspace, i.e. the algebraic structure are related with the permutation symmetry of the qubit-photon coupling terms for the two qubits. Even more interestingly, the quasi-exact solution with at most 1 photon exists in the whole coupling regime with constant eigenenergy equal to single photon energy $\hbar\omega$, which can be clearly found from the algebraic structure. These eigenstates are partly like "dark states", but are coupling dependent and the photon number is bounded from above, so they may have some interesting. According to the algebraic structure

of the two-qubit quantum Rabi model, we may conjecture there are similar "dark states"–like solutions to those models with homogenous qubits-photon coupling terms. For example, we consider the two-qubit Jaynes-Cummings model [35], which is commonly applied for simplicity in the weak coupling regime [38]. Very interestingly, under similar condition, we find many "dark states"–like eigenstates, existing in the whole coupling regime with constant eigenenergy $E = N\hbar\omega$ $(N = -1, 0, 1, \ldots)$, one of which is also the eigenstate of the two-qubit Rabi model. Since the Jaynes-Cumming model is simper than the Rabi model, these eigenstates may provide some interesting application easier. On the other hand, we analytically retrieve the solution of the two-qubit quantum Rabi model, using Bogoliubov operators. With this more physical and straightforward method, we find a way to obtain its solution by convergent power series, so that we can make reasonable cutoff in practical calculation and the solution can reach arbitrary accuracy.

The paper is organized as follows. In section 2, we clarify the algebraic structure behind the possibility of quasi-exact solutions with finite photon numbers obtained in [19] and also find the special "dark states"–like solutions of the two-qubit Jaynes-Cummings model. In section 3, we analytically retrieve the solution of the two-qubit quantum Rabi model using Bogoliubov operators. Finally, we make some conclusions in section 4.

2. Algebraic structure for quasi-exact solutions with finite photon numbers

The Hamiltonian of the two-qubit quantum Rabi model reads ($\hbar = 1$) [17, 19]

$$H_{tq} = \omega a^\dagger a + g_1\sigma_{1x}(a + a^\dagger) + g_2\sigma_{2x}(a + a^\dagger) + \Delta_1\sigma_{1z} + \Delta_2\sigma_{2z}, \tag{1}$$

where $a^\dagger$ and a are the single mode photon creation and annihilation operators with frequency ω, respectively. $\sigma_i\,(i = x, y, z)$ are the Pauli matrices. $2\Delta_1$, $2\Delta_2$ are the energy level splittings of the two qubits. g_1 and g_2 are the qubit-photon coupling constants for the two qubits respectively. There are quasi-exact solutions with finite photon numbers N obtained by analyzing the recurrence relation of the coefficients in [19]. However, the algebraic structure behind the possibility of these novel exceptional solutions needs to be clarified.

Quasi-exact solutions with finite photon numbers N correspond to the existence of closed subspace in the photon number representation, i.e. the algebraic structure. Here we demonstrate the closed subspace are related with the permutation symmetry of the qubit-photon coupling terms by analyzing the structure of the Hamiltonian in the photon number space. H_{tq} (1) process a $\mathbb{Z}_2$ symmetry with the transformation $R = \exp(i\pi a^\dagger a) \otimes \sigma_{1z} \otimes \sigma_{2z}$. Taking odd parity for example, supposing the initial state $|\psi\rangle$ is in a subspace formed by $\{|M, e, g\rangle, |M, g, e\rangle, |M+1, g, g\rangle, |M+1, e, e\rangle, \cdots, |N-1, g, g\rangle, |N-1, e, e\rangle, |N, e, g\rangle,$ $|N, g, e\rangle\}$, with the coefficient $\{c_{1,M}, c_{2,M}, c_{1,M+1}, c_{2,M+1}, \ldots, c_{1,N}, c_{2,N}\}$, where M and N are even, then the Hamiltonian reads (ω is set to 1)

$$\begin{pmatrix} 0 & 0 & \sqrt{M}g_1 & \sqrt{M}g_2 & 0 & 0 & \ldots \\ 0 & 0 & \sqrt{M}g_2 & \sqrt{M}g_1 & 0 & 0 & \ldots \\ \sqrt{M}g_1 & \sqrt{M}g_2 & M + \Delta_1 - \Delta_2 & 0 & \sqrt{M+1}g_1 & \sqrt{M+1}g_2 & \ldots \\ \sqrt{M}g_2 & \sqrt{M}g_1 & 0 & M + \Delta_2 - \Delta_1 & \sqrt{M+1}g_2 & \sqrt{M+1}g_1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

$$
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \sqrt{N}g_1 & \sqrt{N}g_2 & N+\Delta_1-\Delta_2 & 0 & \sqrt{N+1}g_1 & \sqrt{N+1}g_2 \\
\cdots & \sqrt{N}g_2 & \sqrt{N}g_1 & 0 & N+\Delta_2-\Delta_1 & \sqrt{N+1}g_2 & \sqrt{N+1}g_1 \\
\cdots & 0 & 0 & \sqrt{N+1}g_1 & \sqrt{N+1}g_2 & 0 & 0 \\
\cdots & 0 & 0 & \sqrt{N+1}g_2 & \sqrt{N+1}g_1 & 0 & 0
\end{pmatrix}. \tag{2}
$$

If for $|\psi'\rangle = H|\psi\rangle$, the coefficients of $\{|N+1,g,g\rangle,\ |N+1,e,e\rangle\}$ and $\{|M-1,g,g\rangle,\ |M-1,e,e\rangle\}$ equal to 0, then this subspace is closed. For the first case, we obtain

$$\sqrt{N+1}g_1c_{1,N} + \sqrt{N+1}g_2c_{2,N} = 0, \tag{3}$$

$$\sqrt{N+1}g_2c_{1,N} + \sqrt{N+1}g_1c_{2,N} = 0, \tag{4}$$

where $c_{1,N}$ and $c_{2,N}$ are the coefficients of $|N,e,g\rangle$ and $|N,g,e\rangle$ respectively. From equations (3) and (4) and $g_1,\ g_2 > 0$, we obtain $g_1 = g_2$ and $c_{1,N} = -c_{2,N}$. By using the time-independent Schödinger equation, we obtain

$$\sqrt{N}g_1c_{1,N-1} + \sqrt{N}g_2c_{2,N-1} + (N+\Delta_1-\Delta_2)c_{1,N} = Ec_{1,N}, \tag{5}$$

$$\sqrt{N}g_2c_{1,N-1} + \sqrt{N}g_1c_{2,N-1} + (N+\Delta_2-\Delta_1)c_{2,N} = Ec_{2,N}, \tag{6}$$

so that

$$E = N, \tag{7}$$

$$(\Delta_2-\Delta_1)c_{1,N} = (\sqrt{N}g_1c_{1,N-1} + \sqrt{N}g_2c_{2,N-1}). \tag{8}$$

For the special case $\Delta_1 = \Delta_2$ and $c_{1,N-1} = c_{2,N-1} = 0$, there is a invariant subspace formed by $\{|N,e,g\rangle, |N,g,e\rangle\}$, and the eigenstate is

$$|\psi\rangle_N = \frac{1}{\sqrt{2}}(|N,g,e\rangle - |N,e,g\rangle), \tag{9}$$

which is the famous “dark state” [19, 39], where the spin singlet is decoupled from the photon field.

If $\Delta_1 \neq \Delta_2$, considering the coefficient of $\{|M-1,g,g\rangle,\ |M-1,e,e\rangle\}$ must be 0, it is required that $E = M$, which contradicts with $E = N$, so that the only possible choice is $M = 0$. Now we have obtained a closed subspace (algebraic structure) formed by $\{|0,e,g\rangle, |0,g,e\rangle, |1,g,g\rangle, |1,e,e\rangle, \cdots, |N-1,g,g\rangle, |N-1,e,e\rangle, |N,e,g\rangle, |N,g,e\rangle\}$, with the condition

$$g_1 = g_2, \tag{10}$$

$$E = N. \tag{11}$$

Then by using the time-independent Schödinger equation, we can obtain quasi-exact solutions with finite photon number N for certain choice of parameters Δ_1, Δ_2, and $g = g_1 + g_2$. For example, if $N = 2$, the determinant of the matrix

$$
\begin{pmatrix}
\Delta_1-\Delta_2-2 & 0 & g/2 & g/2 & 0 & 0 \\
0 & \Delta_2-\Delta_1-2 & g/2 & g/2 & 0 & 0 \\
g/2 & g/2 & -\Delta_1-\Delta_2-1 & 0 & \sqrt{2}g/2 & \sqrt{2}g/2 \\
g/2 & g/2 & 0 & \Delta_2+\Delta_1-1 & \sqrt{2}g/2 & \sqrt{2}g/2 \\
0 & 0 & \sqrt{2}g/2 & \sqrt{2}g/2 & \Delta_1-\Delta_2 & 0 \\
0 & 0 & \sqrt{2}g/2 & \sqrt{2}g/2 & 0 & \Delta_2-\Delta_1
\end{pmatrix} \tag{12}
$$

must equal to 0, which gives

$$(\Delta_1^2-\Delta_2^2)[(4-(\Delta_1-\Delta_2)^2)(1-(\Delta_1+\Delta_2)^2)-2g^2]=0. \tag{13}$$

This is the condition for an odd parity solution with photon number bounded from above at $N=2$, coinciding with [19], which depends on Δ_1, Δ_2 and g. So now, we have found the algebraic structure and quasi-exact solutions with finite photon numbers N. Furthermore, it is very interesting for the solution with $N=1$, whose existing condition is independent of g. The closed subspace is formed by $\{|0,e,g\rangle,|0,g,e\rangle,|1,g,g\rangle,|1,e,e\rangle\}$, and the condition is

$$\det\begin{vmatrix} \Delta_1-\Delta_2-1 & 0 & g/2 & g/2 \\ 0 & \Delta_2-\Delta_1-1 & g/2 & g/2 \\ g/2 & g/2 & -\Delta_1-\Delta_2 & 0 \\ g/2 & g/2 & 0 & \Delta_2+\Delta_1 \end{vmatrix}=0, \tag{14}$$

which gives

$$(\Delta_1+\Delta_2)^2[(\Delta_1-\Delta_2)^2-1]=0, \tag{15}$$

which is independent of g, coinciding with [19]. So for $\Delta_1-\Delta_2=1=\hbar\omega$ and $\Delta_1-\Delta_2=-1=-\hbar\omega$, we obtain two quasi-exact solutions

$$|\psi\rangle_{g1}=\frac{1}{\mathcal{N}}\left(\frac{2(\Delta_1+\Delta_2)}{g}|0,e,g\rangle+|1,g,g\rangle-|1,e,e\rangle\right), \tag{16}$$

$$|\psi\rangle_{g2}=\frac{1}{\mathcal{N}}\left(\frac{2(\Delta_1+\Delta_2)}{g}|0,g,e\rangle+|1,g,g\rangle-|1,e,e\rangle\right), \tag{17}$$

respectively, where $\mathcal{N}=\sqrt{4(\Delta_1+\Delta_2)^2+2g^2}/g$. For example, choosing $\Delta_1=1.4$, $\Delta_2=0.4$, $g_1=g_2$, the numerical spectrum of the two-qubit quantum Rabi model is shown in figure 1. The horizontal line at $E=1=\hbar\omega$ corresponds to the special eigenstate $|\psi\rangle_{g1}$ (equation (16)). This eigenstate exists in the whole coupling regime with constant eigenenergy, like “dark states”, but are coupling dependent, and with at most 1 photon.

For even parity, similarly, we obtain one such special eigenstate

$$|\psi\rangle_{e}=\frac{1}{\mathcal{N}}\left(\frac{2(\Delta_1-\Delta_2)}{g}|0,e,e\rangle-|1,e,g\rangle+|1,g,e\rangle\right), \tag{18}$$

with the condition $\Delta_1+\Delta_2=1=\omega$, $g_1=g_2$ and $E=1=\hbar\omega$, consistent with [19]. Now, we have demonstrated all the exceptional eigenstates of the two-qubit quantum Rabi model with finite photon numbers presented in [19] by finding its algebraic structure in the photon number space.

The special eigenstates $|\psi\rangle_{g1}$, $|\psi\rangle_{g2}$ and $|\psi\rangle_e$ originate from the permutation symmetry of the qubit-photon coupling terms, and we may conjecture there are similar solutions for similar models. In the weak-coupling regime, the Rabi model can reduce to Jaynes-Cummings model by the rotating-wave approximation, so if there are similar special eigenstates for the two-qubit Jaynes-Cunmmings model, we may find its application in a simpler way. Now we try to find similar structure for the two-qubit Jaynes-Cunmmings model [38]

$$H_{tjc}=a^\dagger a+g_1(\sigma_1^+a+\sigma_1^-a^\dagger)+g_2(\sigma_2^+a+\sigma_2^-a^\dagger)+\Delta_1\sigma_{1z}+\Delta_2\sigma_{2z}. \tag{19}$$

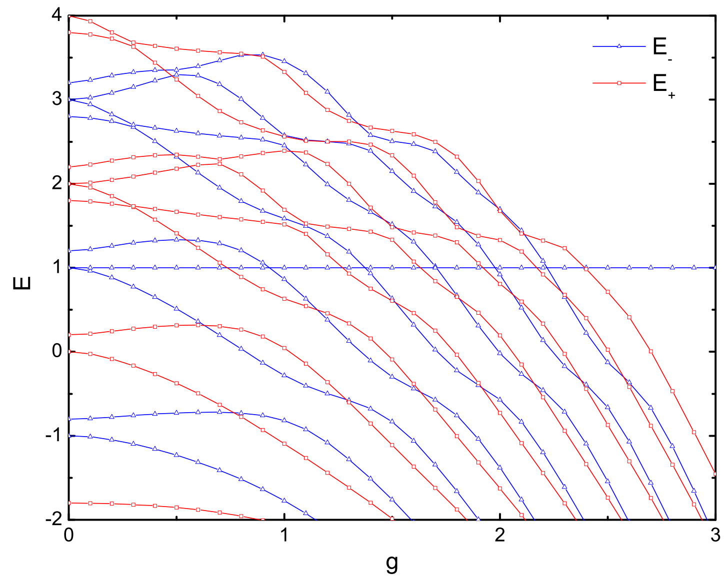

Figure 1. The numerical spectrum of two-qubit quantum Rabi model with $\Delta_1 = 1.4$, $\Delta_2 = 0.4$, $\omega = 1$, $g_1 = g2$, $0 \leq g = g_1 + g_2 \leq 3$. E_+ and E_- are solutions with even and odd parity respectively.

It is easy to find $C = a^\dagger a + \frac{1}{2}(\sigma_{1z} + \sigma_{2z} + 2)$ commutes with H_{tjc}, so there is a conserved quantity C. Interestingly, $|\psi\rangle_e$ (equation (18)) has a conserved quantity $C = 2$, and it is easy to testify $|\psi\rangle_e$ is also an eigenstate of H_{tjc} existing in the whole coupling regime with constant eigenenergy for $g_1 = g_2$ and $\Delta_1 + \Delta_2 = 1$. To find out all such kinds of eigenstates, we study the eigenproblem of H_{tjc}. For $C = N$ $(N > 1)$, the Hamiltonian in the subspace $\{|N-2, e, e\rangle, |N-1, e, g\rangle, |N-1, g, e\rangle, |N, g, g\rangle\}$ reads

$$\begin{pmatrix} N-2+\Delta_1+\Delta_2 & \sqrt{N-1}g_2 & \sqrt{N-1}g_1 & 0 \\ \sqrt{N-1}g_2 & N-1+\Delta_1-\Delta_2 & 0 & \sqrt{N}g_1 \\ \sqrt{N-1}g_1 & 0 & N-1-\Delta_1+\Delta_2 & \sqrt{N}g_2 \\ 0 & \sqrt{N}g_1 & \sqrt{N}g_2 & N-\Delta_1-\Delta_2 \end{pmatrix}. \tag{20}$$

Using the time-independent Schödinger equation, we find the eigenvalues E is determined by

$$\begin{aligned} &(E-N+\Delta_1+\Delta_2)(E-N+2-\Delta_1-\Delta_2)[(E-N+1)^2-(\Delta_1-\Delta_2)^2] \\ &+(g_1^2+g_2^2)[(E-N+1)(E-N+\Delta_1+\Delta_2)-2N(E-N+1)^2] \\ &+(g_1^2-g_2^2)[(g_1^2-g_2^2)(N^2-N)+(\Delta_1^2-\Delta_2^2)(2N-1)+(E+N)(\Delta_2-\Delta_1)]=0. \end{aligned} \tag{21}$$

The condition (equation (21)) is generally dependent on g_1 and g_2, but there are two special cases. The first is the famous "dark state" $|\psi\rangle = \frac{1}{\sqrt{2}}(|N-1, e, g\rangle - |N-1, g, e\rangle$, with the condition $g_1 = g_2$ and $\Delta_1 = \Delta_2$. The spin singlet is decoupled from the photon field, so the eigenenergy and eigenstate are coupling-independent. The second case is partly like "dark state"—the eigenenergy is also coupling independent, but the eigenstate is not. For $g_1 = g_2$

and $\Delta_1 + \Delta_2 = 1$, equation (21) reduces to

$$(E - N + 1)^2[(E - N + 1)^2 + (\frac{1}{2} - N)g^2 - (\Delta_1 - \Delta_2)^2] = 0, \tag{22}$$

where $g = g_1 + g_2$. For $E = N - 1$, the condition is g-independent. Besides, the eigenenergies are symmetric about $E = N - 1$ and there are two degenerate eigenstates with $E = N - 1$ existing in the whole coupling regime

$$|\psi_{C=N_a}\rangle = \frac{1}{\mathcal{A}}\left(\frac{2(\Delta_1 - \Delta_2)}{\sqrt{N-1}g}|N-2, e, e\rangle - |N-1, e, g\rangle + |N-1, g, e\rangle\right), \tag{23}$$

$$\begin{aligned} |\psi_{C=N_b}\rangle = & \frac{1}{\mathcal{B}}\left(\frac{\sqrt{N-1}g}{(\Delta_1 - \Delta_2)}|N-2, e, e\rangle + |N-1, e, g\rangle - |N-1, g, e\rangle\right. \\ & \left. + \frac{(N-1)g^2 + 2(\Delta_1 - \Delta_2)^2}{\sqrt{N}g(\Delta_2 - \Delta_1)}|N, g, g\rangle\right), \end{aligned} \tag{24}$$

where $\frac{1}{\mathcal{A}}$ and $\frac{1}{\mathcal{B}}$ are the normalizing constants. For $N = 2$, $|\psi_{C=2_a}\rangle$ is just $|\psi\rangle_e$ (equation (17)), which is the eigenstate of the two-qubit quantum Rabi model.

For $C = 1$, the subspace is formed by $\{|0, e, g\rangle, |0, g, e\rangle, |1, g, g\rangle\}$, and the eigenvalues satisfy

$$\begin{aligned} & E[(\Delta_1 - \Delta_2)^2 + E(1 - \Delta_1 - \Delta_2) - E^2 + g_1^2 + g_2^2] \\ & \quad + (\Delta_1 + \Delta_2 - 1)(\Delta_1 - \Delta_2)^2 + (g_1^2 - g_2^2)(\Delta_1 - \Delta_2) = 0. \end{aligned} \tag{25}$$

For $g_1 = g_2$ and $\Delta_1 + \Delta_2 = 1$, equation (25) reduces to

$$E[E^2 - \frac{1}{2}g^2 - (\Delta_1 - \Delta_2)^2] = 0. \tag{26}$$

So there is an eigenstate existing in the whole coupling regime with constant eigenenergy $E = 0$

$$|\psi_{C=0}\rangle = \frac{1}{\mathcal{N}}\left(\frac{2(\Delta_1 - \Delta_2)}{g}|1, g, g\rangle - |0, e, g\rangle + |0, g, e\rangle\right). \tag{27}$$

For $C = 0$, the eigenstate is $|0, g, g\rangle$, with constant eigenenergy $E = -1$.

To conclude, for identical-coupling $g_1 = g_2$ and quasi-resonant condition $\Delta_1 + \Delta_2 = 1 = \omega$, the spectrum of the two-qubit Jaynes-Cummings Hamiltonian H_{tjc} is very regular and interesting: there are horizontal lines at $E = N$ ($N = -1, 0, 1, \ldots$), and the energy curve with the same $C = N$ are symmetric about the line $E = N - 1$. For $C = 0, 1$, there is one kind of eigenstates existing in the whole coupling regime with constant eigenenergy, while for other cases, there are two such kinds of degenerate eigenstaes, one of which for $C = 2$ is also the eigenstate of the two-qubit Rabi model. With constant eigenenergy, these eigenstates are partly like "dark state", but they are coupling dependent. Choosing $\Delta_1 = 0.7$, $\Delta_2 = 0.3$ and $g_1 = g_2 = g/2$, the spectra of the two-qubit Jaynes-Cummings model and Rabi model are compared in figure 2.

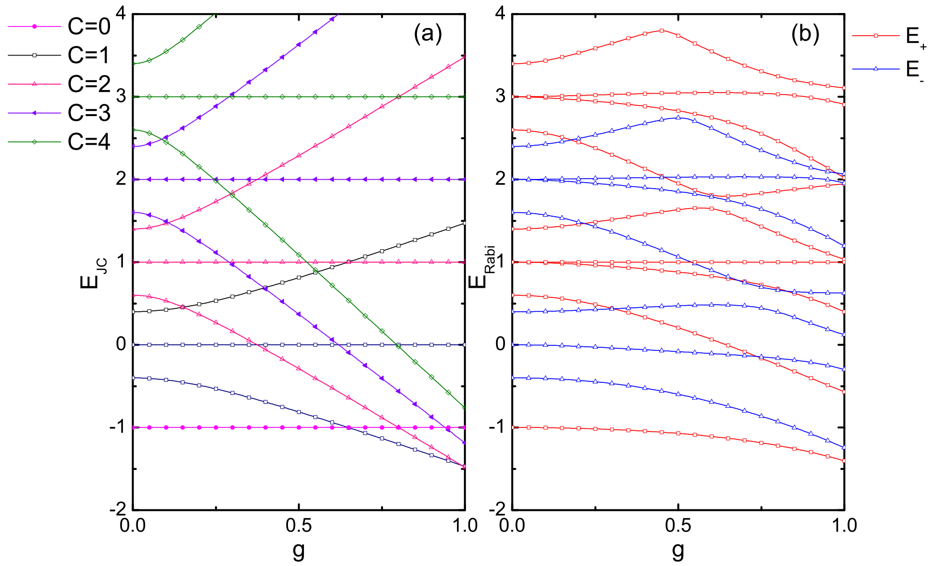

Figure 2. (a) The spectrum of two-qubit Jaynes-Cummings model with $\Delta_1 = 0.7$, $\Delta_2 = 0.3$, $\omega = 1$, $g_1 = g_2$, $0 \leq g = g_1 + g_2 \leq 1$. (b) The numerical spectrum of two-qubit quantum Rabi model with the same parameters. E_+ and E_- are solutions with even and odd parity respectively.

3. Solvability of the two-qubit quantum Rabi model using Bogoliubov operators

First for convenient, we make unitary transformations $S_1 = \frac{1}{\sqrt{2}}(\sigma_{1x} + \sigma_{1z})$ and $S_2 = \frac{1}{\sqrt{2}}(\sigma_{2x} + \sigma_{2z})$ to the two-qubit Rabi Hamiltonian (equation (1)) to obtain (ω is set to 1)

$$H'_{tq} = a^\dagger a + g_1\sigma_{1z}(a + a^\dagger) + g_2\sigma_{2z}(a + a^\dagger) + \Delta_1\sigma_{1x} + \Delta_2\sigma_{2x}. \tag{28}$$

H'_{tq} has a conserved parity with the $\mathbb{Z}_2$ transformation $R = T \otimes \sigma_{1x} \otimes \sigma_{2x}$, where $T = \exp(i\pi a^\dagger a)$, giving us a way to diagonalize the Hamiltonian in the basis of $\{|e\rangle_1, |g\rangle_1\}$, which is the eigenvector of σ_{1z}. Applying the Fulton-Gouterman transformation [16, 40],

$$U = \begin{pmatrix} 1 & 1 \\ T \otimes \sigma_{2x} & -T \otimes \sigma_{2x} \end{pmatrix}, \tag{29}$$

we obtain

$$U^\dagger H'_{tq} U = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{30}$$

where

$$H_\pm = a^\dagger a + g_1(a + a^\dagger) + g_2(a + a^\dagger)\sigma_{2z} + \Delta_2\sigma_{2x} \pm \Delta_1 T\sigma_{2x}, \tag{31}$$

acting on the subspace of R with eigenvalues ± 1. First we consider H_+. For H_-, we just need to substitute $-\Delta_1$ for Δ_1. In the basis of $\{|e\rangle_2 \otimes |\phi_1\rangle, |g\rangle_2 \otimes |\phi_2\rangle\}$, where $|\phi_1\rangle$ and $|\phi_2\rangle$

are photon field states, H_+ is expanded as

$$\begin{pmatrix} a^\dagger a + g(a+a^\dagger) & \Delta_2 + \Delta_1 T \\ \Delta_2 + \Delta_1 T & a^\dagger a + g'(a+a^\dagger) \end{pmatrix}. \tag{32}$$

To remove the linear terms of $a^\dagger$ and a, we use the following Bogoliubov operators

$$A = a + g,\ B = a + g'. \tag{33}$$

Firstly we use the Bogoliubov operator A. The time-independent schördinger equation reads

$$(A^\dagger A - g^2 - E)|\phi_1\rangle + \Delta_2|\phi_2\rangle + \Delta_1|\phi_4\rangle = 0, \tag{34}$$

$$[A^\dagger A + (g'-g)(A+A^\dagger) + g^2 - 2gg' - E]|\phi_2\rangle + \Delta_2|\phi_1\rangle + \Delta_1|\phi_3\rangle = 0, \tag{35}$$

where $|\phi_3\rangle = T|\phi_1\rangle$, $|\phi_4\rangle = T|\phi_2\rangle$. To apply the reflection symmetry, we make the transformation T to (34) and (35) to obtain

$$(A^\dagger A - 2g(A+A^\dagger) + 3g^2 - E)|\phi_3\rangle + \Delta_2|\phi_4\rangle + \Delta_1|\phi_2\rangle = 0, \tag{36}$$
$$[A^\dagger A - (g'+g)(A+A^\dagger) + g^2 + 2gg' - E]|\phi_4\rangle + \Delta_2|\phi_3\rangle + \Delta_1|\phi_1\rangle = 0. \tag{37}$$

We expand the photon field states $|\phi_j\rangle,\ j = 1,\ \ldots,\ 4$ in terms of the normalized orthogonal extended coherent state [12]

$$|n, g\rangle = \frac{e^{-g^2/2 - ga^\dagger}}{\sqrt{n!}}(a^\dagger + g)^n, \tag{38}$$

which is the eigenstate of $A^\dagger A$, and obtain

$$|\phi_j\rangle = e^{g^2/2}\sum_{n=0}^{\infty}\sqrt{n!}a_{j,n}|n, g\rangle,\ j = 1, \ldots, 4. \tag{39}$$

Substituting (39) into (34)–(37), and left multiply $\langle m, g|$, we obtain the recurrence relations for $a_{j,m}$

$$(E - m + g^2)a_{1,m} = \Delta_1 \quad a_{4,m} + \Delta_2 a_{2,m}, \tag{40}$$
$$\begin{aligned}(g'-g)(m+1)a_{2,m+1} = {} & (E - m + 2gg' - g^2)a_{2,m} - (g'-g)a_{2,m-1} \\ & - \Delta_1 a_{3,m} - \Delta_2 a_{1,m},\end{aligned} \tag{41}$$
$$2g(m+1)a_{3,m+1} = (m - E + 3g^2)a_{3,m} - 2ga_{3,m-1} + \Delta_1 a_{2,m} + \Delta_2 a_{4,m}, \tag{42}$$
$$\begin{aligned}(g+g')(m+1)a_{4,m+1} = {} & (m - E + 2gg' + g^2)a_{4,m} - (g+g')a_{4,m-1} \\ & + \Delta_1 a_{1,m} + \Delta_2 a_{3,m}.\end{aligned} \tag{43}$$

It is seen the coefficients $a_{j,m}$ depend on three initial conditions, which can be chosen as $\{a_{1,0},\ a_{2,0},\ a_{3,0}\}$.

Then we consider the Bogoliubov operator $B = a + g'$. Now H_+ is given as

$$\begin{pmatrix} B^\dagger B + (g-g')(B+B^\dagger) + (g')^2 - 2gg' & \Delta_2 + \Delta_1 T \\ \Delta_2 + \Delta_1 T & B^\dagger B - g' \end{pmatrix}. \tag{44}$$

Applying transformation T to the time-independent schödinger equation, we obtain four equations similar to (34)–(37)

$$(B^\dagger B+(g-g')(B+B^\dagger)+(g')^2-2gg'-E)|\varphi_1\rangle+\Delta_2|\varphi_2\rangle+\Delta_1|\varphi_4\rangle=0, \tag{45}$$

$$(B^\dagger B-(g')^2-E)|\varphi_2\rangle+\Delta_2|\varphi_1\rangle+\Delta_1|\varphi_3\rangle=0, \tag{46}$$

$$(B^\dagger B-(g'+g)(B+B^\dagger)+(g')^2+2gg'-E)|\varphi_3\rangle+\Delta_2|\varphi_4\rangle+\Delta_1|\varphi_2\rangle=0, \tag{47}$$

$$(B^\dagger B-2g'(B+B^\dagger)+3(g')^2-E)|\varphi_4\rangle+\Delta_2|\varphi_3\rangle+\Delta_1|\varphi_1\rangle=0. \tag{48}$$

Expanding the photon field states as $|\varphi_j\rangle=e^{(g')^2/2}\sum_{n=0}^{\infty}\sqrt{n!}b_{j,n}|n,g'\rangle,\ j=1,\ldots,4$, where the normalized extended coherent state $|n,g'\rangle$ is the eigenstate of B, and left multiplying $\langle m,g'|$, we obtain the recurrence relations for $b_{j,m}$

$$\begin{aligned}(g-g')(m+1)b_{1,m+1}=&\,(E-m+2gg'-(g')^2)b_{1,m}\\&+(g'-g)b_{1,m-1}-\Delta_1 b_{4,m}-\Delta_2 b_{2,m},\end{aligned} \tag{49}$$

$$(E-m+(g')^2)b_{2,m}=\Delta_1 b_{3,m}+\Delta_2 b_{1,m}, \tag{50}$$

$$\begin{aligned}(g+g')(m+1)b_{3,m+1}=&\,(m-E+2gg'+(g')^2)b_{3,m}\\&-(g+g')b_{3,m-1}+\Delta_1 b_{2,m}+\Delta_2 b_{4,m},\end{aligned} \tag{51}$$

$$\begin{aligned}2g'(m+1)b_{4,m+1}=&\,(m-E+3(g')^2)b_{4,m}\\&-2g'b_{4,m-1}+\Delta_1 b_{1,m}+\Delta_2 b_{3,m}.\end{aligned} \tag{52}$$

There are three initial conditions, which can be chosen as $\{b_{1,0},\ b_{2,0},\ b_{4,0}\}$. To utilize the reflection symmetry $|\phi_1\rangle=T|\phi_3\rangle$, $|\phi_2\rangle=T|\phi_4\rangle$, finally, we expand the photon states in terms of the photon number states as $|\psi_j\rangle=\sqrt{n!}c_{j,n}|n\rangle$, and obtain the recurrence relations for $c_{j,m}$

$$(m+1)gc_{1,m+1}=(E-m)c_{1,m}-gc_{1,m-1}-\Delta_2 c_{2,m}-\Delta_1 c_{4,m}, \tag{53}$$

$$(m+1)g'c_{2,m+1}=(E-m)c_{2,m}-g'c_{2,m-1}-\Delta_2 c_{1,m}-\Delta_1 c_{3,m}, \tag{54}$$

$$(m+1)gc_{3,m+1}=(m-E)c_{3,m}-gc_{3,m-1}+\Delta_2 c_{4,m}+\Delta_1 c_{2,m}, \tag{55}$$

$$(m+1)g'c_{4,m+1}=(m-E)c_{4,m}-g'c_{4,m-1}+\Delta_2 c_{3,m}+\Delta_1 c_{1,m}. \tag{56}$$

Considering $|\psi_1\rangle=T|\psi_3\rangle$, $|\psi_2\rangle=T|\psi_4\rangle$, we obtain $c_{1,m}=(-1)^m c_{3,m}$, $c_{2,m}=(-1)^m c_{4,m}$, so there are only two initial conditions, which can be chosen as $c_{1,0}$ and $c_{2,0}$.

States $|\phi_j\rangle$, $|\varphi_j\rangle$ and $|\psi_j\rangle$ in different representations should be only different by a constant (here can be chosen as 1) if they are nondegenerate eigenstates with eigenvalue E, so we obtain 8 equations

$$|\phi_j\rangle=|\varphi_j\rangle, \tag{57}$$

$$|\varphi_j\rangle=|\psi_j\rangle. \tag{58}$$

For practical calculation, we left multiply $\langle 0|e^{\beta}a$, where β is chosen arbitrarily, then (57) and (58) are mapped to

$$\begin{aligned}\langle 0|e^{\beta_1 a}|\phi_j\rangle&=\sum_{m=0}^{\infty}a_{j,m}\exp(-g\beta_1)(\beta_1+g)^m\\&=\langle 0|e^{\beta_1 a}|\psi_j\rangle=\sum_{m=0}^{\infty}b_{j,m}\exp(-g'\beta_1)(\beta_1+g')^m,\end{aligned} \tag{59}$$

$$\langle 0|e^{\beta_2 a}|\varphi_j\rangle = \sum_{m=0}^{\infty} b_{j,m} \exp(-g'\beta_2)(\beta_2 + g')^m$$
$$= \langle 0|e^{\beta_2 a}|\psi_j\rangle = \sum_{m=0}^{\infty} c_{j,m}\beta_2^m. \quad (60)$$

Now we are still dealing with power series with infinite terms, so to obtain clear reliable result, we must make all the power series convergent. According to the recurrence relations for $a_{j,m}$ (equations (40)–(43)), $b_{j,m}$ (equations (49)–(52)) and $c_{j,m}$ (equations (53)–(56)), we find the radii of convergence of corresponding power series are $|g - g'|$, $\min\{g - g', 2g'\}$ and g' respectively. So, for different g and g', we can always choose proper β_1 and β_2 to obtain convergent power series [19], so that finite terms can give reliable results and by choosing proper cutoff, and we can obtain the results with arbitrary accuracy. That is the advantage of choosing these three different representations. Because of the linearity of recurrence relations, we can denote

$$\phi_j(\beta_1) = \langle 0|e^{\beta_1 a}|\phi_j\rangle = \sum_{k=1}^{3} a_{k,0}\phi_j^k(\beta_1), \quad (61)$$
$$\varphi_j(\beta_1) = \langle 0|e^{\beta_1 a}|\varphi_j\rangle = \sum_{k=1,2,4} b_{k,0}\varphi_j^k(\beta_1), \quad (62)$$
$$\varphi_j(\beta_2) = \langle 0|e^{\beta_2 a}|\varphi_j\rangle = \sum_{k=1,2,4} b_{k,0}\varphi_j^k(\beta_2), \quad (63)$$
$$\psi_j(\beta_2) = \langle 0|e^{\beta_2 a}|\psi_j\rangle = \sum_{k=1}^{2} c_{k,0}\psi_j^k(\beta_2), \quad (64)$$

where for example, $\varphi_j^k(\beta_1)$ is obtained by setting $b_{k,0}$ equal to 1 and other initial conditions equal to 0 in equations (49)–(52), like in [23]. Now we have eight initial conditions for eight equations

$$\phi_j(\beta_1) = \varphi_j(\beta_1), \quad (65)$$
$$\varphi_j(\beta_2) = \psi_j(\beta_2), \quad (66)$$

which can be denoted as

$$M_{jk}e_k = 0, \quad (67)$$

with $\vec{e} = \{b_{1,0}, b_{2,0}, b_{4,0}, a_{1,0}, a_{2,0}, a_{3,0}, c_{1,0}, c_{2,0}\}^T$. The determinant of M, which is just the function of energy E must equal to 0, so we obtain

$$G_+(E) = det(M_+) = 0, \quad (68)$$

which can be used to determine the eigenenergy E. Equation (68) takes similar form as equation (14) in [19], but are obtained in a simper and more physical way. Choosing $\Delta_1 = 0.4$, $\Delta_2 = 0.3$, $\omega = 1$, $g_1 = 3g_2$, to have convergent power series in equations (59) and (60), we can choose $\beta_1 = -3g_2$ and $\beta_2 = -g_2$, then the results can be obtained with arbitrary accuracy. The spectrum is shown in figure 3. It is seen there are no level crossings within the same parity subspace, so we can label each eigenstates with two quantum numbers—energy level

and parity, but the total degrees of freedom are three, so according to the quantum integrability criterion proposed by Braak [9], the model is non-integrable, consistent with what the narrow avoided crossings in the same parity subspace indicate [41] and the result in [19].

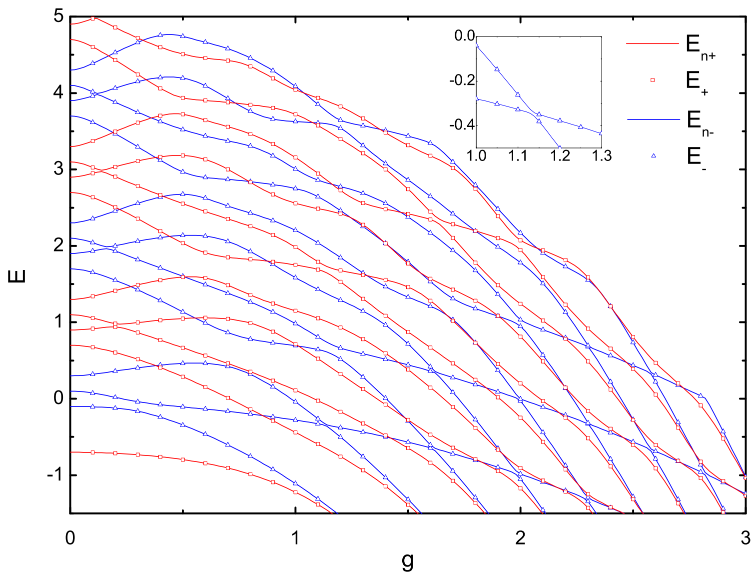

Figure 3. The spectrum of two-qubit quantum Rabi model with $\Delta_1 = 0.4$, $\Delta_2 = 0.3$, $\omega = 1$, $g_1 = 3g_2$, $0 \leq g = g_1 + g_2 \leq 3$. E_{n+} and E_{n-} are numerical solutions with even and odd parity respectively, while E_+ and E_- are analytical solutions with even and odd parity respectively.

4. Conclusions

We have clarified the algebraic structure behind the possibility of the quasi-exact solutions with finite photon numbers found in [19]. By analyzing the Hamiltonian structure in the photon number space, we find that the permutation symmetry of the qubit-photon coupling terms for the two qubits brings about closed subspace, and hence quasi-exact solutions for certain parameters. The novel coupling-dependent eigenstates existing in the whole coupling regime with constant eigenenergy E equal to single photon energy $\hbar\omega$ correspond to quasi-exact solutions with at most 1 photon, with the condition for the qubits energy splittings $\Delta_1 \pm \Delta_2 = \hbar\omega$ or $\Delta_2 - \Delta_1 = \hbar\omega$. We have demonstrated this directly from the Hamiltonian structure. These special eigenstates are partly like "dark states", but are coupling-dependent, which may have some potential application. Furthermore, based on our study on the two-qubit quantum Rabi model, we conjecture such "dark states"-like eigenstates commonly exist in similar models with permutation symmetry of the qubit-photon coupling terms. For example, for the homogenous coupled two-qubit Jaynes-Cummings model, there are many such kinds of eigenstates with constant energy $E = N\hbar\omega$ $(N = -1, 0, 1, \ldots)$ in the whole coupling regime, with the condition $\Delta_1 + \Delta_2 = \hbar\omega$. One of these special states is also the eigenstate

of the two-qubit quantum Rabi model. Since the Jaynes-Cummings model is simper than the Rabi model, we may find the application of these special eigenstates easier.

Besides, using Bogoliubov operators, we have analytically retrieved the solution of the two-qubit quantum Rabi model. We find three different representations to expand the Hamiltonian, and the solutions can be determined by convergent power series. In this way, the eigenproblem of the infinite dimensional Hamiltonian can reduces to finite dimensional in practical calculation reasonably, and the results can reach arbitrary accuracy. Without using Bargmann space, this method is more physical and concise.

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